Classification of random processes:-

(i) If $t \rightarrow$ Continuous & $X \rightarrow$ Continuous

$x(t) \rightarrow$ Continuous random process

(ii) If $t \rightarrow$ Continuous & $X \rightarrow$ discrete

$x(t) \rightarrow$ discrete random process

(iii) If $t \rightarrow$ discrete & $X \rightarrow$ Continuous

$x(t) \rightarrow$ Continuous random sequence

(iv) If $t \rightarrow$ discrete & $X \rightarrow$ discrete

$x(t) \rightarrow$ discrete random sequence.

Deterministic & Non-deterministic random process:-

(i) A R.P is deterministic if the future values of any sample function can be predicted from the past values.

(ii) A R.P is non-deterministic if the future values of any sample function cannot be predicted exactly from observed past values.

# Distribution and density function :-

Let $X(t)$ be a random process. For a fixed time $t_1$, $X(t_1)$ becomes a random variable. $(X_1)$

The distribution function or cummulative distribution fn of the R.V $X(t_1)$ is denoted by $F(x_1, t_1)$ & is defined by

$$\boxed{F(x_1, t_1) = P[X(t_1) \le x_1] \quad \forall\, x_1}$$

Also, the density fn. of $X(t)$ is given by

$$\boxed{f(x_1, t_1) = \frac{\partial F(x_1, t_1)}{\partial x_1}}$$

The above fns $F(x_1, t_1)$ and $f(x_1, t_1)$ are called first order distribution and first order density of the process $X(t)$.

IIIly

2nd order distribution is $\boxed{F(x_1, x_2; t_1, t_2) = P[X(t_1) \le x_1, X(t_2) \le x_2]}$

2nd order density is $\boxed{f(x_1, x_2; t_1, t_2) = \frac{\partial^2 F(x_1, x_2; t_1, t_2)}{\partial x_1 \partial x_2}}$

Remark:-

1. The above definitions can be extended upto $n^{th}$ order for $n$ RV's $x(t_1)\ X(t_2) \ldots X(t_n)$.

2. $F_X(x_1, t_1) = F(x_1, \infty; t_1, t_2)$

   $F_X(x_2, t_1) = F(\infty, x_2; t_1, t_2)$

3. $f_X(x_1, t_1) = \int_{-\infty}^{\infty} f(x_1, x_2; t_1, t_2)\, dx_2$

   $f_X(x_2, t_2) = \int_{-\infty}^{\infty} f(x_1, x_2; t_1, t_2)\, dx_1$

Statistical Averages:-

The random process can be described by statistical averages given below.

(i) Mean / Mean function / ensemble average:

The mean of the R.P $X(t)$ is the expected value of the random variable $X(t)$ at any time $t$.

(ie) $\mu_X(t) = E[X(t)] = \int_{-\infty}^{\infty} x\, f(x, t)\, dx \qquad -\infty < t < \infty$

② Autocorrelation fn:- $R_{XX}(t_1, t_2)$ :- $[R_X(t_1, t_2)]$ ⑤

The autocorrelation fn. $R_{XX}(t_1, t_2)$ is defined as,

$$R_{XX}(t_1, t_2) = E\left[X(t_1)X(t_2)\right]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f(x_1, x_2, t_1, t_2)\, dx_1\, dx_2.$$

where $X(t_1)$ & $X(t_2)$ are two random variables for any two times $t_1, t_2$.

Remark:-

① $R_{XX}(t_1, t_1) = E\left[X(t_1)\ X(t_1)\right]$

$= E\left[X^2(t_1)\right]$

$=$ 2nd moment

③ Auto Covariance :- $C_{XX}(t_1, t_2)$ / $C_X(t_1, t_2)$ / $C(t_1, t_2)$

$$C_{XX}(t_1, t_2) = E\left\{\left[X(t_1) - \mu(t_1)\right]\left[X(t_2) - \mu(t_2)\right]\right\}$$

$$= R_{XX}(t_1, t_2) - \mu(t_1)\,\mu(t_2)$$

Remark :-

$$C_{XX}(t_1,t_1) = R_{XX}(t_1,t_1) - \mu(t_1)\,\mu(t_1)$$

$$= E[X^2] - \{E[X]\}^2$$

$$= Var(X) \qquad X = X(t_1)$$

$$= Var[X(t_1)]$$

④ Correlation Coefficient :-

$$\rho_{XX}(t_1,t_2) = \frac{C_{XX}(t_1,t_2)}{\sqrt{C_{XX}(t_1,t_2)}\,\sqrt{C_{XX}(t_2,t_2)}}$$

# Stationary Process:

## First order stationary process:

A random process $X(t)$ is said to be stationary to order one or $1^{st}$ order stationary if its first order density fn. does not change with a shift in time origin (ie)

$$f_X(x_1; t_1) = f_X(x_1; t_1 + \delta)$$ for any time $t_1$ & any real no. $\delta$.

## Second order stationary process:

A random process $X(t)$ is said to be stationary to order 2 or $2^{nd}$ order stationary if its second order density fn. does not change with a shift in time origin.

$$f_X(x_1, x_2; t_1, t_2) = f_X(x_1, x_2; t_1+\delta, t_2+\delta)$$
$$\forall\ t_1, t_2\ \&\ \delta$$

## $n^{th}$ order stationary process:

$$f_X(x_1, x_2 \ldots x_n; t_1 t_2 \ldots t_n) = f_X(x_1, x_2 \ldots x_n, t_1+\delta, t_2+\delta \ldots t_n+\delta)$$
$$\forall\ t_i\ \&\ \delta$$

Evolutionary process or Non-stationary process:

A random process that is not stationary in any sense is called an evolutionary process or non-stationary process.

Wide-sense stationary process :- (WSS)

A random process $X(t)$ is said to be WSS if the following conditions are satisfied

(i) $E[X(t)] = \mu_X$ (ie) mean is constant

(ii) $E[X(t)\, X(t+\tau)] = R_{XX}(\tau)$

(ie) the autocorrelation fun. depends only on the time difference.

Remark :-

WSS is also called as weak sense stationary process or Covariance stationary process.

Strict sense stationary process: [SSS]

A random process $X(t)$ is said to be stationary in the strict sense (or) strict sense stationary if all its statistical properties are invariant to a shift of time origin.

Remark:-

① It is also known as strongly stationary process.

② Another defn. of SSS :-

A random process $X(t)$ is said to be strict sense stationary if $X(t)$ is stationary to all orders.

(ie) $f_X(x_1, x_2 \dots x_n; t_1 t_2 \dots t_n) = f_X(x_1 x_2 \dots x_n; t_1+\delta, t_2+\delta \dots t_n+\delta)$

$\forall \delta$

$\& \forall\ n \geq 1.$

① Prove that a 1st order stationary random process has a constant mean.

Let $X(t)$ be a random process which is stationary to 1st order.

(ie) $f_X(x,t) = f_X(x, t+\delta)$ for any $\delta$.

To prove: $E[X(t)] = E[X(t+\delta)]$ for any $\delta$.

$$E[X(t)] = \int_{-\infty}^{\infty} x \, f_X(x,t) \, dx$$

$$= \int_{-\infty}^{\infty} x \, f_X(x, t+\delta) \, dx$$

$$= E[X(t+\delta)]$$

$\Rightarrow$ $E[X(t)]$ is a constant.

② Prove that for a second order stationary process, the autocorrelation function is a function of time difference.

Proof:

Let $X(t)$ be a $2^{nd}$ order stationary process.

Let $X(t) = X_1$ & $X(t+\tau) = X_2$.

Given $X(t)$ is of $2^{nd}$ order stationary $\Rightarrow$

$$f_X(x_1, x_2, t_1, t_2) = f_X(x_1, x_2, t_1+\delta, t_2+\delta)$$

To prove: $R_{XX}(t, t+\tau) = R_{XX}(\tau)$

$$R_{XX}(t, t+\tau) = E[X(t)\, X(t+\tau)]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2\, f_X(x_1, x_2, t, t+\tau)\, dx_1\, dx_2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2\, f_X(x_1, x_2, t+\delta, t+\tau+\delta)\, dx_1\, dx_2$$

$+\delta$

In particular put $\delta = -t$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2\, f_X(x_1, x_2, 0, \tau)\, dx_1\, dx_2$$

$$= E[X(0)\, X(\tau)]$$

$$= R_{XX}(0, \tau)$$

$$= R_{XX}(\tau)$$

$$\begin{array}{ccccc} \text{1st order stationary} & \Longleftarrow & \text{2nd order stationary} & \Longrightarrow & \text{WSS} \\ & & & & \Uparrow \\ & & & & \text{SSS} \end{array}$$

Covariance and Toeplitz.

Properties of auto correlation function of atleast WSS process :-



[ Given $X(t)$ is a WSS process

$\Rightarrow$ (i) $E[X(t)] =$ Constant

(ii) $R_{XX}[t, t+\tau] = R_{XX}(\tau)$

Also W.K.T $R_{XX}(t, t+\tau) = E[X(t)\ X(t+\tau)]$ ]

① P.T $R_{XX}$ is an even function

(ie) $R_{XX}(\tau) = R_{XX}(-\tau)$.

Proof :-

$R_{XX}(\tau) = R_{XX}[t, t+\tau]$

$= E[X(t)\ X(t+\tau)]$

$|||^{ly}$ $R_{XX}(-\tau) = E[X(t)\ X(t-\tau)]$ put $t-\tau = t_1$

$= E[X(t_1+\tau)\ X(t_1)]$

$= R_{XX}(t_1, t_1+\tau)$

$= R_{XX}(\tau)$.

(2) P.T $|R_{xx}(\tau)| \leq R_{xx}(0)$



Proof :-

$$R_{xx}(\tau) = R_{xx}(t, t+\tau)$$

$$= E[x(t)\ x(t+\tau)]$$

$$\therefore R_{xx}(0) = E[x(t)\ x(t)]$$

$$= E[x^2(t)] \quad \forall\ t$$

Consider,

$$E[x(t) + x(t+\tau)]^2 \geq 0$$

$$E[x^2(t) + x^2(t+\tau) + 2\,x(t)\,x(t+\tau)] \geq 0$$

$$E[x^2(t)] + E[x^2(t+\tau)] + 2\,E[x(t)\,x(t+\tau)] \geq 0$$

$$R_{xx}(0) + R_{xx}(0) + 2\,R_{xx}(\tau) \geq 0$$

$$2R_{xx}(0) + 2\,R_{xx}(\tau) \geq 0$$

$$R_{xx}(\tau) \geq -R_{xx}(0)$$

$|||^{ly}$ if $E[x(t) - x(t+\tau)]^2 \geq 0$

$$\Rightarrow R_{xx}(\tau) \leq R_{xx}(0)$$

$$\therefore -R_{xx}(0) \leq R_{xx}(\tau) \leq R_{xx}(0)$$

3 If the process $x(t)$ is periodic $R_{xx}(\tau)$ is also periodic with the same period.

Given $x(t)$ is periodic

$\Rightarrow$ $x(t) = x(t+T)$ where $T$ is the period.

To prove :- $R_{xx}(\tau) = R_{xx}(\tau + T)$

$$R_{xx}(\tau) = E[x(t)\ x(t+\tau)]$$
$$= E[x(t)\ x(t+T+\tau)]$$
$$= E[x(t)\ x(t+\{\tau+T\})]$$
$$= R_{xx}(\tau+T)$$

Hence proved.

④ If $x(t)$ is a stationary process with no periodic component then $\lim_{\tau\to\infty} R_{xx}(\tau) = \mu_x^2$, if the limit exists (ie) $\mu_x = \sqrt{\lim_{\tau\to\infty} R_{xx}(\tau)}$.

WKT $R_{xx}(\tau) = E[x(t)\, x(t+\tau)]$

As $\lim_{\tau\to\infty} R_{xx}(\tau) = E[x(t)]\, E[x(t+\tau)] \rightarrow ①$

$= \mu_x\, \mu_x \rightarrow ②$

$= \mu_x^2$

$\Rightarrow \mu_x = \sqrt{\lim_{\tau\to\infty} R_{xx}(\tau)}$

① → As $\tau\to\infty$, $\tau$ is very large (ie) $x(t)$ & $x(t+\tau)$ are two sample fn. observed over a long time period.

∴ $x(t)$ & $x(t+\tau)$ become independent.

[ If dependent, $x(t)$ becomes periodic ]

② → As $x(t)$ is stationary, $E[x(t)] = \mu_x$ (constant)

① Find the mean and variance of the poisson process.

Proof:-

WKT $P[X(t)=n] = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$ $n=0,1,2\ldots$

Mean $= E[X(t)]$

$= \sum_{n=0}^{\infty} n\, P[X(t)=n]$

$= \sum_{n=0}^{\infty} n \frac{e^{-\lambda t}(\lambda t)^n}{n!}$

$= e^{-\lambda t} \sum_{n=1}^{\infty} \frac{(\lambda t)^n}{(n-1)!}$

$= (\lambda t)\, e^{-\lambda t} \sum_{n=1}^{\infty} \frac{(\lambda t)^{n-1}}{(n-1)!}$

$e^{x} = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$= (\lambda t)\, e^{-\lambda t} e^{\lambda t}$

$= \lambda t$

Consider

$E[X^2(t)] = \sum_{n=0}^{\infty} n^2 \frac{e^{-\lambda t}(\lambda t)^n}{n!}$

$= \sum_{n=0}^{\infty} n(n-1) \frac{e^{-\lambda t}(\lambda t)^n}{n!}$

$+ \sum_{n=0}^{\infty} n \frac{e^{-\lambda t}(\lambda t)^n}{n!}$

$$= e^{-\lambda t} \sum_{n=2}^{\infty} \frac{(\lambda t)^n}{(n-2)!} + e^{-\lambda t} \sum_{n=1}^{\infty} \frac{(\lambda t)^n}{(n-1)!}$$

$$= e^{-\lambda t} (\lambda t)^2 \sum_{n=2}^{\infty} \frac{(\lambda t)^{n-2}}{(n-2)!} + e^{-\lambda t} (\lambda t) \sum_{n=1}^{\infty} \frac{(\lambda t)^{n-1}}{(n-1)!}$$

$$= e^{-\lambda t} (\lambda t)^2 e^{\lambda t} + e^{-\lambda t} (\lambda t) e^{\lambda t}$$

$$= (\lambda t)^2 + (\lambda t)$$

$$\therefore Var[X(t)] = E[X^2(t)] - \{E[X(t)]\}^2$$

$$= (\lambda t)^2 + \lambda t - (\lambda t)^2$$

$$= \lambda t.$$

2. Find the autocorrelation fn/. of the poisson random process.

Proof:-

W.K.T $P[X(t) = n] = \frac{e^{-\lambda t} (\lambda t)^n}{n!}$ $n = 0, 1, 2 \ldots$

Consider, $R_{XX}(t_1, t_2) = E[X(t_1) X(t_2)]$

$= E[x(t_1)x(t_2) - x(t_1)x(t_1) + x(t_1)x(t_1)]$

$= E[x(t_1)[x(t_2) - x(t_1)] + x^2(t_1)]$

$= E\{x(t_1)[x(t_2) - x(t_1)]\} + E\{x^2(t_1)\}$

[If $t_2 > t_1$, the intervals $(0, t_1)$ and $(t_1, t_2)$ are non-overlapping & hence $x(t_1)$ & $x(t_2) - x(t_1)$ are independent]

$= E\{x(t_1)\} E\{x(t_2) - x(t_1)\} + E\{x^2(t_1)\}$

$= \lambda t_1 [\lambda t_2 - \lambda t_1] + \lambda^2 t_1^2 + \lambda t_1$

$= \lambda^2 t_1 t_2 - \lambda^2 t_1^2 + \lambda^2 t_1^2 + \lambda t_1$

$= \lambda^2 t_1 t_2 + \lambda t_1$

Remark:-

If $t_1 > t_2$,

$R_{xx}(t_1, t_2) = \lambda^2 t_1 t_2 + \lambda t_2$

(3) P.T sum of two independent poisson process is a poisson process.

let $X_1(t)$ & $X_2(t)$ be two independent poisson processes with rate $\lambda_1$ and $\lambda_2$.

W.K.T

$$P[X_1(t) = n_1] = \frac{e^{-\lambda_1 t} (\lambda_1 t)^{n_1}}{n_1!} \quad n_1 = 0, 1, 2 \dots$$

$$\therefore MGF[X_1(t)] = M_{X_1(t)}(s) = E[e^{s X_1(t)}]$$

$$= \sum_{n_1=0}^{\infty} e^{s n_1} \frac{e^{-\lambda_1 t} (\lambda_1 t)^{n_1}}{n_1!}$$

$$= e^{-\lambda_1 t} \sum_{n=0}^{\infty} \frac{(e^{s} \lambda_1 t)^{n_1}}{n_1!}$$

$$= e^{-\lambda_1 t} e^{e^{s} \lambda_1 t}$$

$$= e^{\lambda_1 t (e^{s} - 1)}$$

$|||^{ly}$ $M_{X_2(t)}(s) = e^{\lambda_2 t (e^{s} - 1)}$

Since $X_1(t)$ and $X_2(t)$ are independent, 

$$M_{X_1(t)+X_2(t)}(s) = M_{X_1(t)}(s) \; M_{X_2(t)}(s)$$

$$= e^{\lambda_1 t(e^s-1)} \; e^{\lambda_2 t(e^s-1)}$$

$$= e^{t(e^s-1)(\lambda_1+\lambda_2)}$$

$\Rightarrow X_1(t) + X_2(t)$ is a poisson process with parameter $\lambda_1+\lambda_2$.

④ Prove that the difference of two independent poisson process is not a poisson process.

Proof :-

Let $X_1(t)$ and $X_2(t)$ be two poisson processes with parameters $\lambda_1$ & $\lambda_2$.

Let $X(t) = X_1(t) - X_2(t)$

$$E[X(t)] = E[X_1(t) - X_2(t)]$$

$$= \lambda_1 t - \lambda_2 t$$

$$= (\lambda_1 - \lambda_2)t$$

Consider,

$E[X^2(t)] = E[X_1(t) - X_2(t)]^2$ (6)

$= E[X_1^2(t)] - 2E[X_1(t)X_2(t)] + E[X_2^2(t)]$

$= \lambda_1^2 t^2 + \lambda_1 t - 2E[X_1(t)]E[X_2(t)] + \lambda_2^2 t^2 + \lambda_2 t$

$= \lambda_1^2 t^2 + \lambda_1 t - 2\lambda_1 t \lambda_2 t + \lambda_2^2 t^2 + \lambda_2 t$

$= t^2(\lambda_1 - \lambda_2)^2 + t(\lambda_1 + \lambda_2)$

$\neq t^2(\lambda_1 - \lambda_2)^2 + t(\lambda_1 - \lambda_2)$

$\Rightarrow$ Diff is not poisson.

(5)
The interarrival time of a poisson process with parameter $\lambda$ follows exponential distribution with para $\lambda$ & mean $1/\lambda$.

Proof :-

Let $X(t)$ be a poisson process with parameter $\lambda$.

$A_i$ & $A_{i+1}$ occur at two consecutive time intervals say $t$ & $t+T$. Then the (7)

inter arrival time $T$ is a continuous R.V, $t \geq 0$

Consider cdf of $T(t)$,

$$F(t) = P[T \leq t]$$

$$= 1 - P[T > t]$$

$$= 1 - P[\text{No event occurred in } (0, t)]$$

$$= 1 - P[X(t) = 0]$$

$$= 1 - e^{-\lambda t}$$

Now pdf of $T(t)$,

$$f(t) = F'(t) = \lambda e^{-\lambda t} \quad \lambda > 0, \; t \geq 0$$

which is the density fn. of expo. distribution.

$\Rightarrow$ Mean $= 1/\lambda$.

(6) If the no of occurrences of an event (8)
A in an interval (0,t) is a poisson process
X(t) with parameter $\lambda$ and if each
occurrences of A has a constant prob p
of being recorded & the recordings are
independent of each other then the no
N(t) of recorded occurrences in t is also
a poisson process with para $\lambda p$.

Let X(t) be a poisson process with
parameter $\lambda$.

$$\Rightarrow P[X(t) = r] = \frac{e^{-\lambda t} (\lambda t)^r}{r!} \quad r = 0,1,2 \ldots$$

Suppose of the $n+r$ occurrences in t,
n of them are recorded,

$$P[N(t) = n] = \sum_{r=0}^{\infty} P[\text{A occurs } n+r \text{ times in } t \text{ and } n \text{ of them are recorded}]$$

$$= \sum_{r=0}^{\infty} P[A \text{ occurs } n+r \text{ times}]$$ 9

$$P[n \text{ recorded out of } n+r]$$

$$= \sum_{r=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^{n+r}}{(n+r)!} \, {}^{(n+r)}C_n \, p^n q^{(n+r)-n}$$

$$= \sum_{r=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^{n}(\lambda t)^{r}}{(n+r)!} \, {}^{(n+r)}C_r \, p^n q^r$$

$$= \sum_{r=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^{n}}{\cancel{(n+r)!}} \frac{\cancel{(n+r)!}}{r!\,(n)!} p^n q^r (\lambda t)^r$$

$$= \frac{e^{-\lambda t}(\lambda t)^n p^n}{n!} \sum_{r=0}^{\infty} \frac{(q\lambda t)^r}{r!}$$

$$= \frac{e^{-\lambda t}(\lambda t p)^n}{n!} e^{q\lambda t}$$

$$= \frac{e^{-\lambda t(1-q)}(\lambda t p)^n}{n!} = \frac{e^{-\lambda t p}(\lambda t p)^n}{n!}$$

$\Rightarrow$ $N(t)$ is also poisson with para $\lambda p$.

7. If $N_1(t)$ and $N_2(t)$ denotes independent poisson process with parameters $\lambda_1$ and $\lambda_2$ respectively, prove that conditional prob of $N_1(t)$ given $N_1(t)+N_2(t)$ follows binomial distribution.

Proof:-

Given $N_1(t)$ and $N_2(t)$ represent Poisson process,

$$P[N_1(t)=n] = \frac{e^{-\lambda_1 t}(\lambda_1 t)^n}{n!}$$

$$P[N_2(t)=n] = \frac{e^{-\lambda_2 t}(\lambda_2 t)^n}{n!}$$

WKT when $N_1(t)$ and $N_2(t)$ follow Poisson process their sum $N_1(t)+N_2(t)$ also follows Poisson process with parameter $\lambda_1+\lambda_2$.

$$\Rightarrow P[N_1(t)+N_2(t)=n] = \frac{e^{-(\lambda_1+\lambda_2)t}[(\lambda_1+\lambda_2)t]^n}{n!}$$

Consider $P\left[\dfrac{N_1(t)=m}{N_1(t)+N_2(t)=n}\right]$

$$= \frac{P[N_1(t)=m,\ N_1(t)+N_2(t)=n]}{P[N_1(t)+N_2(t)=n]}$$

$$= \frac{P[N_1(t)=m,\ N_2(t)=n-m]}{P[N_1(t)+N_2(t)=n]}$$

$$= \frac{P[N_1(t)=m]\ P[N_2(t)=n-m]}{P[N_1(t)+N_2(t)=n]} \quad \left[\because N_1(t) \text{ and } N_2(t) \text{ are independent}\right]$$

$$= \frac{\cancel{e^{-\lambda_1 t}}(\lambda_1 t)^m}{m!}\ \frac{\cancel{e^{-\lambda_2 t}}(\lambda_2 t)^{n-m}}{(n-m)!}\ \frac{n!}{\cancel{e^{-(\lambda_1+\lambda_2)t}}\left[(\lambda_1+\lambda_2)t\right]^n}$$

$$= \frac{n!}{m!\,(n-m)!}\ \frac{(\lambda_1 t)^m\,(\lambda_2 t)^{n-m}}{\left[(\lambda_1+\lambda_2)t\right]^{n-m+m}}$$

$$= \frac{n!}{m!\,(n-m)!}\ \frac{(\lambda_1 t)^m}{\left[(\lambda_1+\lambda_2)t\right]^m}\ \frac{(\lambda_2 t)^{n-m}}{\left[(\lambda_1+\lambda_2)t\right]^{n-m}}$$

→ The no of customers serviced per unit time is called the service rate & is denoted by $\mu$.

→ The service time is a Random Variable and assumed to follow exponential distribution with mean rate of service $\mu$.

## ③ Service Channels:

→ Service channels may be arranged in parallel or in series or both depending on the design of the model.

→ By Parallel channel we mean a no of channels providing identical service facilities so that several customers may be serviced simultaneously.

→ In series channel, a customer must pass successively thro' all the ordered channels before service is completed.

4. Queue discipline :-

→ Rule according to which customers are selected for service when a queue has been formed.

→ Common disciplines are,

FIFO / FCFS

LIFO / LCFS

SIRO

Priority

5. Capacity of the system :-

Maximum no of customers in the system can be either finite or infinite.

6. Customer Behaviour :-

→ Balking

→ Reneging

→ Jockeying

## State of the system :-

The states of the queueing system are classified as

(i) Transient state

(ii) Steady state

(iii) Explosive state.

### (i) Transient state :

→ Operating characteristics [ input, output, mean queue length ... ] are dependent on time.

→ Usually occur at the beginning of the queueing system.

### (ii) Steady state :

→ Operating characteristics are independent of time.

→ $\lim_{t \to \infty} P_n(t) = P_n$

& $\lim_{t \to \infty} P_n'(t) = 0$

(iii) Explosive state:

If $\lambda > \mu$ then queue length tends to $\infty$. This state is said to be explosive state.

Kendal's notation for representing queueing models:

$$(a/b/c):(d/e)$$

a → arrival probability distribution

b → service time probability distribution.

[ M → Markov / Poisson / Exponential

$E_k$ → Erlangian / Gamma

G → General ]

c → no of servers

d → capacity of the system

e → queue discipline.

## Notations Used:-



$P_n \rightarrow$ Prob of n customers

$P_0 \rightarrow$ Prob of zero (or) no customers

$L_s \rightarrow$ Avg. no of customers in system

$L_q \rightarrow$ Avg. no of customers in queue

$W_s \rightarrow$ Avg. waiting time in system

$W_q \rightarrow$ Avg. waiting time in queue

Little's formula :- [Relation between $L_s, L_q, W_s, W_q$]

$$L_s = \lambda W_s$$

$$L_q = \lambda W_q$$

$$L_s = L_q + \frac{\lambda}{\mu}$$

$$W_q = W_s - \frac{1}{\mu}$$

Birth and death process:

Consider a Markovian queue. Let

$P_n(t) \rightarrow$ Prob of $n$ customers in the queueing system at time $t$, $n > 0$.

$\lambda_n \rightarrow$ average arrival rate when there are $n$ customers in the system.

$\mu_n \rightarrow$ average service rate when there are $n$ customers in the system.

Then,

$P[1 \text{ arrival in time } \Delta t] = \lambda_n \Delta t$

$P[\text{No arrival in time } \Delta t] = 1 - \lambda_n \Delta t$

$P[\text{More than 1 arrival in } \Delta t] = 0$

$P[1 \text{ service in time } \Delta t] = \mu_n \Delta t$

$P[\text{No service in } \Delta t] = 1 - \mu_n \Delta t$

$P[\text{More than 1 service in } \Delta t] = 0$

Case (i) Let us consider the prob of n customers in the system at time $t+\Delta t$.

| time $t$ | Arrival in $\Delta t$ | Service in $\Delta t$ | Time $t+\Delta t$ |
|---|---|---|---|
| $n$ | 0 | 0 | $n$ |
| $n$ | 1 | 1 | $n$ |
| $n-1$ | 1 | 0 | $n$ |
| $n+1$ | 0 | 1 | $n$ |

$$P_n(t+\Delta t) = P_n(t)\, P\begin{bmatrix}\text{No arrival} \\ \text{in } \Delta t\end{bmatrix} P\begin{bmatrix}\text{No service} \\ \text{in } \Delta t\end{bmatrix}$$

$$+ P_n(t)\, P\begin{bmatrix}\text{one arrival} \\ \text{in } \Delta t\end{bmatrix} P\begin{bmatrix}\text{one service} \\ \text{in } \Delta t\end{bmatrix}$$

$$+ P_{n-1}(t)\, P\begin{bmatrix}\text{One arrival} \\ \text{in } \Delta t\end{bmatrix} P\begin{bmatrix}\text{No service} \\ \text{in } \Delta t\end{bmatrix}$$

$$+ P_{n+1}(t)\, P\begin{bmatrix}\text{No arrival} \\ \text{in } \Delta t\end{bmatrix} P\begin{bmatrix}\text{one service} \\ \text{in } \Delta t\end{bmatrix}$$

$$= P_n(t)\,[1-\lambda_n \Delta t]\,[1-\mu_n \Delta t] +$$

$$P_n(t)\,[\lambda_n \Delta t]\,[\mu_n \Delta t] +$$

$$P_{n-1}(t)\,[\lambda_{n-1}\Delta t]\,[1-\mu_{n-1}\Delta t] +$$

$$P_{n+1}(t)\,[1-\lambda_{n+1}\Delta t]\,[\mu_{n+1}\Delta t]$$

$$= P_n(t) - \lambda_n P_n(t)\Delta t - \mu_n P_n(t)\Delta t + P_{n-1}(t)\lambda_{n-1}\Delta t$$
$$+ P_{n+1}(t)\mu_{n+1}\Delta t + o[\Delta t^2]$$

$$P_n(t+\Delta t) - P_n(t) = -\lambda_n P_n(t)\Delta t - \mu_n P_n(t)\Delta t +$$
$$P_{n-1}(t)\lambda_{n-1}\Delta t + P_{n+1}(t)\mu_{n+1}\Delta t + o[\Delta t^2]$$

$$\frac{P_n(t+\Delta t) - P_n(t)}{\Delta t} = -\lambda_n P_n(t) - \mu_n P_n(t) + P_{n-1}(t)\lambda_{n-1}$$
$$+ P_{n+1}(t)\mu_{n+1} + o(\Delta t)$$

$$\lim_{\Delta t \to 0} \frac{P_n(t+\Delta t) - P_n(t)}{\Delta t} = -\lambda_n P_n(t) - \mu_n P_n(t) + P_{n-1}(t)\lambda_{n-1}$$
$$+ P_{n+1}(t)\mu_{n+1}$$

$$P_n'(t) = -\lambda_n P_n(t) - \mu_n P_n(t) + P_{n-1}(t)\lambda_{n-1}$$
$$+ P_{n+1}(t)\mu_{n+1}$$

Under <u>steady state</u> conditions, $P_n(t) \rightarrow P_n$

and $P_n'(t) \rightarrow 0$

$$0 = -\lambda_n P_n - \mu_n P_n + P_{n-1}\lambda_{n-1} + P_{n+1}\mu_{n+1}$$

$$P_n[\lambda_n + \mu_n] = P_{n-1}\lambda_{n-1} + P_{n+1}\mu_{n+1}$$

Case (ii)

Let us consider the prob. of zero customers in the system at time $t+\Delta t$.

| time $t$ | Arrival in $\Delta t$ | Service in $\Delta t$ | time $t+\Delta t$ |
|---|---|---|---|
| 0 | 0 | — | 0 |
| 0 | 1 | 1 | 0 |
| 1 | 0 | 1 | 0 |

$$P_0(t+\Delta t) = P_0(t)\,[1-\lambda_0\,\Delta t] + P_0(t)\,[\lambda_1 \Delta t]\,[\mu_1 \Delta t]$$
$$+ P_1(t)\,[1-\lambda_1\,\Delta t]\,[\mu_1\,\Delta t]$$

$$= P_0(t) - \lambda_0\,P_0(t)\,\Delta t + P_1(t)\,\mu_1\,\Delta t + o[\Delta t^2]$$

$$\frac{P_0(t+\Delta t) - P_0(t)}{\Delta t} = -\lambda_0\,P_0(t) + P_1(t)\,\mu_1 + o[\Delta t]$$

$$\lim_{\Delta t\to 0} \frac{P_0(t+\Delta t) - P_0(t)}{\Delta t} = -\lambda_0\,P_0(t) + \mu_1\,P_1(t)$$

$$P_0'(t) = -\lambda_0\,P_0(t) + \mu_1\,P_1(t)$$

Using steady state conditions

$$0 = -\lambda_0\,P_0 + \mu_1\,P_1$$

$$\lambda_0 P_0 = \mu_1 P_1$$

$$\boxed{P_1 = \frac{\lambda_0}{\mu_1}\,P_0}$$

Sub $n=1$ in equa 1

$$[\lambda_1 + \mu_1] P_1 = P_0 \lambda_0 + P_2 \mu_2$$

$$\lambda_1 P_1 + \mu_1 P_1 - P_0 \lambda_0 = P_2 \mu_2$$

$$P_2 \mu_2 = \lambda_1 \left(\frac{\lambda_0 P_0}{\mu_1}\right) + \mu_1 \left(\frac{\lambda_0 P_0}{\mu_1}\right) - P_0 \lambda_0$$

$$P_2 \mu_2 = \frac{\lambda_1 \lambda_0}{\mu_1} P_0$$

$$\boxed{P_2 = \frac{\lambda_0 \lambda_1}{\mu_1 \mu_2} P_0}$$

|||ly

$$P_3 = \frac{\lambda_0 \lambda_1 \lambda_2}{\mu_1 \mu_2 \mu_3} P_0$$

$\vdots$

$$\boxed{P_n = \frac{\lambda_0 \lambda_1 \lambda_2 \dots \lambda_{n-1}}{\mu_1 \mu_2 \dots \mu_n} P_0}$$

Model: I [M/M/1 : ∞/FIFO]

Let $\lambda_n = \lambda \quad \forall n$

$\mu_n = \mu \quad \forall n$

$$P_n = \frac{\lambda\lambda\cdots\lambda \text{ (n terms)}}{\mu\mu\cdots\mu \text{ (n terms)}} P_0$$

$$P_n = \left(\frac{\lambda}{\mu}\right)^n P_0$$

To find $P_0$ :-

$$\sum_{n=0}^{\infty} P_n = 1$$

$$P_0 + P_1 + P_2 + \cdots = 1$$

$$P_0 + \frac{\lambda}{\mu} P_0 + \left(\frac{\lambda}{\mu}\right)^2 P_0 + \cdots = 1$$

$$P_0 \left[1 + \frac{\lambda}{\mu} + \frac{\lambda^2}{\mu^2} + \cdots \right] = 1$$

$$P_0 \left[1 - \frac{\lambda}{\mu}\right]^{-1} = 1$$

$$\boxed{P_0 = 1 - \frac{\lambda}{\mu}}$$

$$\Rightarrow P_n = \left(\frac{\lambda}{\mu}\right)^n \left[1 - \frac{\lambda}{\mu}\right]$$

Measures of Model I

1. $L_s$ = Average no. of customers in the system

$$= E(n)$$

$$= \sum_{n=0}^{\infty} n P_n$$

$$= P_1 + 2P_2 + 3P_3 + \cdots$$

$$= \left(\frac{\lambda}{\mu}\right) P_0 + 2\left(\frac{\lambda}{\mu}\right)^2 P_0 + 3\left(\frac{\lambda}{\mu}\right)^3 P_0 + \cdots$$

$$= \frac{\lambda}{\mu} P_0 \left[1 + 2\frac{\lambda}{\mu} + 3\left(\frac{\lambda}{\mu}\right)^2 + \cdots\right]$$

$$= \frac{\lambda}{\mu} \left[1 - \frac{\lambda}{\mu}\right] \left[1 - \frac{\lambda}{\mu}\right]^{-2}$$

$$= \frac{\lambda}{\mu} \left[1 - \frac{\lambda}{\mu}\right]^{-1}$$

$$= \frac{\lambda}{\mu} \left[\frac{\mu - \lambda}{\mu}\right]^{-1}$$

$$= \frac{\lambda}{\mu} \frac{\mu}{\mu - \lambda}$$

$$= \frac{\lambda}{\mu - \lambda}$$

2. $W_s$ = Average waiting time in the system

$$= \frac{L_s}{\lambda}$$

$$= \frac{\lambda}{(\mu-\lambda)\lambda}$$

$$= \frac{1}{\mu-\lambda}$$

3. $W_q$ = Average waiting time in the queue

$$= W_s - \frac{1}{\mu}$$

$$= \frac{1}{\mu-\lambda} - \frac{1}{\mu}$$

$$= \frac{\mu-\mu+\lambda}{\mu(\mu-\lambda)}$$

$$= \frac{\lambda}{\mu(\mu-\lambda)}$$

4. $L_q$ = Average no of customers in the queue

$$= \lambda W_q$$

$$= \lambda\left[\frac{\lambda}{\mu(\mu-\lambda)}\right]$$

$$= \frac{\lambda^2}{\mu(\mu-\lambda)}$$

5. $P[\text{channel busy}] = \frac{\lambda}{\mu}$

6. $P[\text{idle system}] = 1 - \frac{\lambda}{\mu}$

7. $P[\text{No of customers in the system} \geq k]$

$= P[N=k] + P[N=k+1] + \cdots$

$= \left(\frac{\lambda}{\mu}\right)^k \left[1-\frac{\lambda}{\mu}\right] + \left(\frac{\lambda}{\mu}\right)^{k+1} \left[1-\frac{\lambda}{\mu}\right] + \cdots$

$= \left[1-\frac{\lambda}{\mu}\right] \left[\frac{\lambda}{\mu}\right]^k \left[1 + \frac{\lambda}{\mu} + \left(\frac{\lambda}{\mu}\right)^2 + \cdots\right]$

$= \left[1-\frac{\lambda}{\mu}\right] \left[\frac{\lambda}{\mu}\right]^k \left[1-\frac{\lambda}{\mu}\right]^{-1}$

$= \left(\frac{\lambda}{\mu}\right)^k.$

8. $P[\text{Waiting time in the system} > t] = e^{-(\mu-\lambda)t}$

Model : II $[M/M/c : \infty/FIFO]$

Here $\lambda_n = \lambda \ \forall \ n$

$$\mu_n = \begin{cases} n\mu & n < c \\ c\mu & n \geq c \end{cases}$$

$$P_n = \begin{cases} \dfrac{\lambda\lambda \cdots \lambda \text{ (n terms)}}{\mu \; 2\mu \; 3\mu \cdots n\mu} P_0 & n < c \\[2ex] \dfrac{\lambda\lambda \cdots \lambda \text{ (n terms)}}{\mu \; 2\mu \cdots (c-1)\mu \; c\mu \; c\mu \cdots c\mu} P_0 & n \geq c \end{cases}$$

$$= \begin{cases} \dfrac{(\lambda/\mu)^n}{n!} P_0 & n < c \\[2ex] \dfrac{\lambda^n}{c! \; \mu^c \; \mu^{n-c} \; c^{n-c}} P_0 & n \geq c \end{cases}$$

$$= \begin{cases} \dfrac{(\lambda/\mu)^n}{n!} P_0 & n < c \\[2ex] \dfrac{1}{c!} \dfrac{1}{c^{n-c}} (\lambda/\mu)^n P_0 & n \geq c \end{cases}$$

To find $P_0$ :-

$$\sum_{n=0}^{\infty} P_n = 1$$

$$\sum_{n=0}^{c-1} P_n + \sum_{n=c}^{\infty} P_n = 1$$

$$\sum_{n=0}^{c-1} \frac{1}{n!} \left(\frac{\lambda}{\mu}\right)^n P_0 + \sum_{n=c}^{\infty} \frac{1}{c!} \frac{1}{c^{n-c}} (\lambda/\mu)^n P_0 = 1$$

$$P_0 \left[ \sum_{n=0}^{c-1} \frac{(\lambda/\mu)^n}{n!} + \frac{1}{c!} \sum_{n=c}^{\infty} \frac{1}{c^{n-c}} \left(\frac{\lambda}{\mu}\right)^n \right] = 1$$

$$P_0 \left[ \sum_{n=0}^{c-1} \frac{(\lambda/\mu)^n}{n!} + \frac{1}{c!} \left\{ (\lambda/\mu)^c + \frac{(\lambda/\mu)^{c+1}}{c} + \frac{(\lambda/\mu)^{c+2}}{c^2} + \cdots \right\} \right] = 1$$

$$P_0 \left[ \sum_{n=0}^{c-1} \frac{(\lambda/\mu)^n}{n!} + \frac{(\lambda/\mu)^c}{c!} \left\{ 1 + \frac{(\lambda/\mu)}{c} + \frac{(\lambda/\mu)^2}{c^2} + \cdots \right\} \right] = 1$$

$$P_0 \left[ \sum_{n=0}^{c-1} \frac{(\lambda/\mu)^n}{n!} + \frac{(\lambda/\mu)^c}{c!} \left\{ 1 + \left(\frac{\lambda}{\mu c}\right) + \left(\frac{\lambda}{\mu c}\right)^2 + \cdots \right\} \right] = 1$$

$$P_0 \left[ \sum_{n=0}^{c-1} \frac{(\lambda/\mu)^n}{n!} + \frac{(\lambda/\mu)^c}{c!} \frac{1}{1 - \frac{\lambda}{\mu c}} \right] = 1$$

$$P_0 = \left[ \sum_{n=0}^{c-1} \left[ \frac{(\lambda/\mu)^n}{n!} \right] + \frac{(\lambda/\mu)^c}{c!} \frac{1}{1 - \frac{\lambda}{\mu c}} \right]^{-1}$$

Properties :-

1. Prob that an arrival has to wait

$$= P[N \geq c]$$

$$= \sum_{n=c}^{\infty} P_n$$

$$= \sum_{n=c}^{\infty} \frac{1}{c!} \frac{1}{c^{n-c}} \left(\frac{\lambda}{\mu}\right)^n P_0$$

$$= \sum_{n=c}^{\infty} \frac{1}{c!} \frac{1}{c^{-c}} \left(\frac{\lambda}{\mu c}\right)^n P_0$$

$$= \frac{c^c}{c!} \sum_{n=c}^{\infty} \left(\frac{\lambda}{\mu c}\right)^n P_0$$

$$= \frac{c^c}{c!} \left[\left(\frac{\lambda}{\mu c}\right)^c P_0 + \left(\frac{\lambda}{\mu c}\right)^{c+1} P_0 + \cdots\right]$$

$$= \frac{c^c}{c!} \left(\frac{\lambda}{\mu c}\right)^c P_0 \left[1 + \left(\frac{\lambda}{\mu c}\right) + \cdots\right]$$

$$= \frac{1}{c!} \left(\frac{\lambda}{\mu}\right)^c P_0 \left[1 - \frac{\lambda}{\mu c}\right]^{-1}$$

$$= \frac{1}{c!} \frac{(\lambda/\mu)^c}{\left[1 - \frac{\lambda}{\mu c}\right]} P_0$$

2. $P[\text{Customer need not wait}] = 1 - P[N \geq c]$

3. $L_q$ = Average no of customers in the queue

$$= \sum_{n-c=0}^{\infty} (n-c) P_n \qquad m = n-c$$

$$= \sum_{m=0}^{\infty} m \, P_{m+c}$$

$$= \sum_{m=0}^{\infty} m \frac{1}{c! \, c^m} (\lambda/\mu)^{m+c} P_0$$

$$= \frac{1}{c!} \left(\frac{\lambda}{\mu}\right)^c P_0 \sum_{m=0}^{\infty} \frac{m}{c^m} (\lambda/\mu)^m$$

$$= \frac{(\lambda/\mu)^c}{c!} P_0 \left[ \frac{\lambda}{\mu c} + 2\left(\frac{\lambda}{\mu c}\right)^2 + 3\left(\frac{\lambda}{\mu c}\right)^3 + \cdots \right]$$

$$= \frac{(\lambda/\mu)^c}{c!} P_0 \left[ 1 - \frac{\lambda}{\mu c} \right]^{-2} \left(\frac{\lambda}{\mu c}\right)$$

$$= \frac{(\lambda/\mu)^{c+1}}{c \, c!} P_0 \left[ 1 - \frac{\lambda}{\mu c} \right]^{-2}$$

$$L_q = \frac{(\lambda/\mu)}{c} \frac{P[N \geq c]}{\left[1 - \frac{\lambda}{\mu c}\right]}$$

$$= \frac{\lambda}{\mu c} \frac{\mu c}{\mu c - \lambda} P[N \geq c]$$

$$\boxed{L_q = \frac{\lambda}{\mu c - \lambda} P[N \geq c]}$$

(4) $L_s = L_q + \frac{\lambda}{\mu}$

(5) $W_q = \frac{L_q}{\lambda}$

(6) $W_s = W_q + \frac{1}{\mu}$

(7) Average no of customers in a non-empty queue

is = $\dfrac{\text{Average no of customers in queue}}{\text{Prob of non-empty queue}}$

$$= \frac{L_q}{P[N \geq c]} = \frac{\lambda}{\mu c - \lambda} P[N \geq c] \frac{1}{P[N \geq c]}$$

$$= \frac{\lambda}{\mu c - \lambda}$$

(8) Mean waiting time in the queue for those who actually wait is $= \dfrac{W_q}{P[N \geq c]}$

$$= \frac{L_q}{\lambda P[N \geq c]}$$

$$= \frac{\lambda}{\mu c - \lambda} \quad \frac{P[N \geq c]}{\lambda P[N \geq c]}$$

$$= \frac{1}{\mu c - \lambda}$$

(9) Efficiency of M/M/c model = $\dfrac{\text{Avg no of customers served}}{\text{Total no of customers}}$

(10) Utilization factor = proportion of time channels will be busy

$$= \frac{\lambda}{\mu c}.$$

Model III $[M/M/1] : [N/FIFO]$

$$\lambda_n = \begin{cases} \lambda & n = 0, 1, 2 \ldots N-1 \\ 0 & n \geq N \end{cases}$$

$$\mu_n = \mu \qquad \forall\, n$$

$$P_n = \frac{\lambda_0 \lambda_1 \ldots \lambda_{n-1}}{\mu_1 \mu_2 \ldots \mu_n} P_0$$

16

$$P_n = \left(\frac{\lambda}{\mu}\right)^n P_0$$

To find $P_0$ :-

$$\sum_{n=0}^{N} P_n = 1$$

$$P_0 + P_1 + \cdots + P_N = 1$$

$$P_0 + \frac{\lambda}{\mu} P_0 + \left(\frac{\lambda}{\mu}\right)^2 P_0 + \cdots + \left(\frac{\lambda}{\mu}\right)^N P_0 = 1$$

$$P_0 \left[ 1 + \frac{\lambda}{\mu} + \left(\frac{\lambda}{\mu}\right)^2 + \cdots + \left(\frac{\lambda}{\mu}\right)^N \right] = 1$$

$$\frac{P_0 \left[ 1 - \left(\frac{\lambda}{\mu}\right)^{N+1} \right]}{1 - \frac{\lambda}{\mu}} = 1$$

$$P_0 = \begin{cases} \dfrac{1 - \frac{\lambda}{\mu}}{1 - \left(\frac{\lambda}{\mu}\right)^{N+1}} & \text{if } \frac{\lambda}{\mu} < 1 \\[2ex] \dfrac{1}{N+1} & \text{if } \frac{\lambda}{\mu} = 1 \end{cases}$$

$$\Rightarrow P_n = \begin{cases} \dfrac{\left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n}{1 - \left(\frac{\lambda}{\mu}\right)^{N+1}} & \text{if } \frac{\lambda}{\mu} < 1 \\[2ex] \dfrac{1}{N+1} & \text{if } \frac{\lambda}{\mu} = 1 \end{cases}$$

Characteristics :-

1. Average no. of customers in the system $= L_s$

$$= \sum_{n=0}^{N} n\, P_n$$

$$= \sum_{n=0}^{N} n \left(\frac{\lambda}{\mu}\right)^n P_0$$

$$= \left(\frac{\lambda}{\mu}\right) P_0 \sum_{n=0}^{N} n \left(\frac{\lambda}{\mu}\right)^{n-1}$$

$$= \left(\frac{\lambda}{\mu}\right) P_0 \sum_{n=0}^{N} \frac{d}{dx}\left[x^n\right]$$

$$= \left(\frac{\lambda}{\mu}\right) P_0 \frac{d}{dx} \sum_{n=0}^{N} x^n$$

$$= \left(\frac{\lambda}{\mu}\right) P_0 \frac{d}{dx}\left[\frac{1-x^{N+1}}{1-x}\right]$$

$$= \left(\frac{\lambda}{\mu}\right) P_0 \left[\frac{(1-x)\left[-(N+1)\,x^N\right] - \left[1-x^{N+1}\right](-1)}{(1-x)^2}\right]$$

$$= \left(\frac{\lambda}{\mu}\right) \frac{P_0}{(1-x)^2}\left[-(N+1)\,x^N + (N+1)\,x^{N+1} + 1 - x^{N+1}\right]$$

$$= \left(\frac{\lambda}{\mu}\right) \frac{P_0}{(1-x)^2}\left[1-(N+1)\,x^N + N\,x^{N+1}\right]$$

$$= \left(\frac{\lambda}{\mu}\right) \frac{\left[1 - \frac{\lambda}{\mu}\right]}{1 - \left(\frac{\lambda}{\mu}\right)^{N+1}} \quad \frac{1}{\left(1 - \frac{\lambda}{\mu}\right)^2} \left[1 - (N+1)\left(\frac{\lambda}{\mu}\right)^N + N\left(\frac{\lambda}{\mu}\right)^{N+1}\right]$$

$$= \frac{(\lambda/\mu)}{\left(1 - \frac{\lambda}{\mu}\right)\left[1 - \left(\frac{\lambda}{\mu}\right)^{N+1}\right]} \left[1 - (N+1)\left(\frac{\lambda}{\mu}\right)^N + N\left(\frac{\lambda}{\mu}\right)^{N+1}\right]$$

If $\frac{\lambda}{\mu} = 1$

$$L_s = \sum_{n=0}^{N} n P_n$$

$$= \sum_{n=0}^{N} n \frac{1}{N+1}$$

$$= \frac{1}{N+1} \sum_{n=0}^{N} n$$

$$= \frac{1}{N+1} \frac{N(N+1)}{2}$$

$$= \frac{N}{2}$$

Remark :-

→ '$\lambda$' we use in the Little's formulae is the average arrival rate throughout.

→ In this model, $\lambda$ is the arrival rate until vacancy is there and zero if system is full. Hence we introduce another concept the overall arrival rate $\lambda'$.

(2) $L_q = \sum_{n=1}^{N} (n-1) P_n$

$$= \sum_{n=1}^{N} n P_n - \sum_{n=1}^{N} P_n$$

$$= \sum_{n=0}^{N} n P_n - \left[\sum_{n=0}^{N} P_n - P_0\right]$$

$$= L_s - [1 - P_0]$$

But by Little's formulae $L_q = L_s - \frac{\lambda'}{\mu}$

$$\Rightarrow \frac{\lambda'}{\mu} = 1 - P_0$$

$$\Rightarrow \lambda' = \mu[1 - P_0]$$

Remark:-

The Little formulae becomes,

$$L_q = L_s - \frac{\lambda'}{\mu}$$

$$W_s = \frac{L_s}{\lambda'}$$

$$W_q = \frac{L_q}{\lambda'}$$

(3) $P\left[\text{that a customer turned away}\right] = P_N$

$$= \left(\frac{\lambda}{\mu}\right)^N P_0$$

Model : $\underline{\text{IV}}$ $\quad [M/M/c : N/FIFO]$

$$\lambda_n = \begin{cases} \lambda & n = 0, 1, 2 \ldots N-1 \\ 0 & n \geq N \end{cases}$$

$$\mu_n = \begin{cases} n\mu & 0 \leq n \leq c-1 \\ c\mu & c \leq n \leq N \end{cases}$$

$$P_n = \frac{\lambda_0 \lambda_1 \ldots \lambda_{n-1}}{\mu_1 \mu_2 \ldots \mu_n} P_0$$

$$= \begin{cases} \dfrac{\lambda \lambda \ldots \lambda \ (n \text{ times})}{1\mu \ 2\mu \cdots n\mu} P_0 \\[2ex] \dfrac{\lambda \lambda \ldots \lambda \ (n \text{ times})}{1\mu \ 2\mu \cdots (c-1)\mu \ c\mu \ c\mu \cdots c\mu} P_0 \end{cases}$$

$$= \begin{cases} \dfrac{(\lambda/\mu)^n}{n!} P_0 & 0 \leq n \leq c-1 \\[2ex] \dfrac{(\lambda/\mu)^n}{c! \ c^{n-c}} P_0 & c \leq n \leq N \\[2ex] 0 & n > N \end{cases}$$

To find $P_0$

$$\sum_{n=0}^{N} P_n = 1$$

$$\sum_{n=0}^{c-1} P_n + \sum_{n=c}^{N} P_n = 1$$

$$\sum_{n=0}^{c-1} \frac{(\lambda/\mu)^n}{n!} P_0 + \sum_{n=c}^{N} \frac{(\lambda/\mu)^n}{c!\, c^{n-c}} P_0 = 1$$

$$P_0 = \left[ \sum_{n=0}^{c-1} \frac{(\lambda/\mu)^n}{n!} + \sum_{n=c}^{N} \frac{(\lambda/\mu)^n}{c!\, c^{n-c}} \right]^{-1}$$

Characteristics :-

1. Average number of customers in the queue $L_q$

$m = n-c$

$$= \sum_{n-c=0}^{N} (n-c)\, P_n$$

$$= \sum_{m=0}^{N} m\, P_{m+c}$$

$$= \sum_{m=0}^{N} \frac{m\, (\lambda/\mu)^{m+c}}{c!\, c^m} P_0$$

$$= \frac{(\lambda/\mu)^c}{c!} P_0 \sum_{m=0}^{N} m \left(\frac{\lambda}{\mu c}\right)^m$$

$$= \frac{(\lambda/\mu)^c}{c!} P_0 \left(\frac{\lambda}{\mu c}\right) \sum_{m=0}^{N} m \left(\frac{\lambda}{\mu c}\right)^{m-1}$$

$$= \frac{(\lambda/\mu)^c}{c!} \left(\frac{\lambda}{\mu c}\right) P_0 \frac{d}{dx} \sum_{m=0}^{N} x^m$$

$$= \frac{(\lambda/\mu)^c}{c!} \left(\frac{\lambda}{\mu c}\right) P_0 \frac{d}{dx} \left[\frac{1-x^{N+1}}{1-x}\right] \quad \text{if } x<1$$

$$= \frac{(\lambda/\mu)^c}{c!} \left(\frac{\lambda}{\mu c}\right) \frac{P_0}{(1-x)^2} \left[(1-x)\left[-(N+1)x^N\right] - \left[1-x^{N+1}\right](-1)\right]$$

$$= \frac{(\lambda/\mu)^c}{c!} \left(\frac{\lambda}{\mu c}\right) \frac{P_0}{(1-x)^2} \left[1 + N x^{N+1} - (N+1)x^N\right]$$

$$= \frac{(\lambda/\mu)^c}{c!} \frac{(\lambda/\mu c)}{\left(1-\frac{\lambda}{\mu c}\right)^2} P_0 \left[1-(N+1)\left(\frac{\lambda}{\mu c}\right)^N + N\left(\frac{\lambda}{\mu c}\right)^{N+1}\right]$$

Effective arrival rate :-

$$L_s = \sum_{n=0}^{N} n P_n$$

$$= \sum_{n=0}^{c-1} n P_n + \sum_{n=c}^{N} n P_n - \sum_{n=c}^{N} c P_n + \sum_{n=c}^{N} c P_n$$

$$= \sum_{n=0}^{c-1} n P_n + \sum_{n=c}^{N} c P_n + \sum_{n=c}^{N} (n-c) P_n$$

$$L_s = L_q + \sum_{n=0}^{c-1} n P_n + c \sum_{n=c}^{N} P_n$$

$$L_s = L_q + \sum_{n=0}^{c-1} n P_n + c \left[ 1 - \sum_{n=0}^{c-1} P_n \right]$$

$$L_s = L_q + c + \sum_{n=0}^{c-1} (n-c) P_n$$

$$\frac{\lambda'}{\mu} = c + \sum_{n=0}^{c-1} (n-c) P_n$$

$$\lambda' = \mu \left[ c - \sum_{n=0}^{c-1} (c-n) P_n \right]$$

Model $\underline{V}$ M/M/$\infty$ [Self Service Model]

$$\lambda_n = \lambda \quad \forall n$$

$$\mu_n = n\mu \quad \forall n$$

$$P_n = \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} P_0$$

$$\boxed{P_n = \frac{(\lambda/\mu)^n}{n!} P_0}$$

To find $P_0$ :-

$$\sum_{n=0}^{\infty} P_n = 1$$

$$P_0 + P_1 + \cdots = 1$$

$$P_0 \sum_{n=0}^{\infty} \frac{(\lambda/\mu)^n}{n!} = 1$$

$$P_0 \; e^{\lambda/\mu} = 1$$

$$P_0 = e^{-\lambda/\mu}$$

$$\Rightarrow \quad P_n = \frac{(\lambda/\mu)^n}{n!} e^{-\lambda/\mu}$$

Measures :-

1. $L_s = \sum_{n=0}^{\infty} n \, P_n$

$$= e^{-\lambda/\mu} \sum_{n=0}^{\infty} n \frac{(\lambda/\mu)^n}{n!}$$

$$= e^{-\lambda/\mu} \sum_{n=0}^{\infty} \frac{(\lambda/\mu)^n}{(n-1)!}$$

$$= e^{-\lambda/\mu} \left(\frac{\lambda}{\mu}\right) \sum_{n=0}^{\infty} \frac{(\lambda/\mu)^{n-1}}{(n-1)!}$$

$$= e^{-\lambda/\mu} \frac{\lambda}{\mu} e^{\lambda/\mu}$$

$$= \frac{\lambda}{\mu}$$

2. $W_s = 0$ ; $W_q = 0$ ; $L_q = 0$ 

Model: $\overline{\text{VI}}$ Machine Interference Model
[M/M/R: k/FIFO]

Here the population from where customers come is finite.

Define,
$$\lambda_n = \begin{cases} (k-n)\lambda & 0 \le n \le k \\ 0 & n > k \end{cases}$$

$$\mu_n = \begin{cases} n\mu & 0 \le n \le R-1 \\ R\mu & R \le n \le k \end{cases}$$

$$P_n = \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} P_0$$

$$= \begin{cases} \dfrac{k\lambda \, (k-1)\lambda \cdots (k-n+1)\lambda}{1\mu \; 2\mu \cdots n\mu} P_0 & 0 \le n \le R-1 \\[2ex] \dfrac{k\lambda \, (k-1)\lambda \cdots (k-n+1)\lambda}{1\mu \; 2\mu \cdots (R-1)\mu \; R\mu \; R\mu \cdots R\mu} P_0 & R \le n \le k \end{cases}$$

$$= \begin{cases} kC_n \, (\lambda/\mu)^n \, P_0 & 0 \le n \le R-1 \\[2ex] kC_n \, \dfrac{n! \, (\lambda/\mu)^n}{R! \; R^{n-R}} P_0 & R \le n \le k \end{cases}$$

To find $P_0$ :-

$$\sum_{n=0}^{K} P_n = 1$$

$$\sum_{n=0}^{R-1} P_n + \sum_{n=R}^{K} P_n = 1$$

$$P_0 = \left[ \sum_{n=0}^{R-1} {}^{K}C_n (\lambda/\mu)^n + \sum_{n=R}^{K} {}^{K}C_n \frac{n!(\lambda/\mu)^n}{R!\ R^{n-R}} \right]^{-1}$$

Measures :-

1. $$L_s = \sum_{n=0}^{K} n P_n$$

2. $$L_q = \sum_{n=R}^{K} (n-R) P_n$$

3. $$\lambda' = \sum_{n=0}^{K} (K-n) \lambda P_n = \sum_{n=0}^{K} K \lambda P_n - \sum_{n=0}^{K} n \lambda P_n$$

$$= K\lambda - \lambda E(n)$$

$$= \lambda [K - E(n)]$$

$$= \lambda [K - L_s]$$

4. $$W_q = \frac{L_q}{\lambda'}$$

5. $$W_s = \frac{L_s}{\lambda'}$$

# Unit V - Non Markovian Queues & Queue Networks

## Non-Markovian Queue:

→ A single server queueing system in which arrivals follow Poisson Process and service time distribution is general.

→ When service time is general the no of customers at any time $t$ is not a Markov process.

→ However we define a Markov chain to find parameters of our interest.

## P-K formulae : [Pollaczek - Khinchine]

[ It is used to find average no of customers in the system (ie) $L_s$].

Let arrivals follow Poisson process with rate of arrival $\lambda$.

Service time follow an arbitrary prob distribution and $T$ be the service time between two departures. (ie) (ie) $E(T) = \frac{1}{\mu}$

$$\sigma^2 = Var(T) = E[T^2] - \{E(T)\}^2$$

Define the Markov chain

$X_n$ = No of customers left behind when the $n^{th}$ customer departs at time $t_n$.

$$\Rightarrow X_{n+1} = \begin{cases} X_n + A - 1 & X_n > 0 \\ A & X_n = 0 \end{cases}$$

where $A$ is the no of customers arriving [Poisson] during the service time $T$ of the $n+1^{th}$ customer.

(ie) $E(A) = E[\lambda T] = \lambda E(T)$

$$E(A^2) = \lambda^2 E(T^2) + \lambda E(T)$$

Let $U(X_n)$ be the unit fn. defined by

$$U(X_n) = \begin{cases} 1 & \text{if } X_n > 0 \\ 0 & \text{if } X_n = 0 \end{cases}$$

Then $\boxed{X_{n+1} = X_n + A - U(X_n)} \rightarrow (1)$

Also $U^2(X_n) = U(X_n)$

$$X_n \, U(X_n) = X_n$$

Suppose that the system is in steady state, then the prob of no of customers in the system is independent of time and so is a constant.

(ie) $E[X_{n+1}] = E[X_n]$ & $E[X_{n+1}^2] = E[X_n^2]$

Take expectation on both sides of ①

$$E[X_{n+1}] = E[X_n + A - U(X_n)]$$

$$E[\cancel{X_{n+1}}] = E[\cancel{X_n}] + E[A] - E[U(X_n)]$$

$$E[U(X_n)] = E(A)$$

Also $E[X_{n+1}^2] = E[X_n + A - U(X_n)]^2$

$$= E\left[X_n^2 + A^2 + U^2(X_n) + 2X_n A - 2AU(X_n) - 2X_n U(X_n)\right]$$

$$= E\left[X_n^2 + A^2 + U(X_n) + 2X_n A - 2AU(X_n) - 2X_n\right]$$

$$E[\cancel{X_{n+1}^2}] = E[\cancel{X_n^2}] + E(A^2) + E[U(X_n)] + 2E(X_n)E(A) - 2E(A)E[U(X_n)] - 2E(X_n)$$

$$0 = E(A^2) + E(A) + 2E(X_n)E(A) - 2[E(U)]^2 - 2E[X_n]$$

$$0 = -2E(X_n)[1 - E(A)] + E(A^2) + E(A) - 2[E(A)]^2$$

$$E(X_n) = \frac{E(A^2) + E(A) - 2[E(A)]^2}{2[1 - E(A)]}$$

$$= \frac{\lambda^2 E(T^2) + \lambda E(T) + \lambda E(T) - 2\lambda^2 [E(T)]^2}{2[1 - \lambda E(T)]}$$

$$= \frac{\lambda^2 E(T^2) + 2\lambda E(T) - 2\lambda^2 [E(T)]^2}{2[1 - \lambda E(T)]}$$

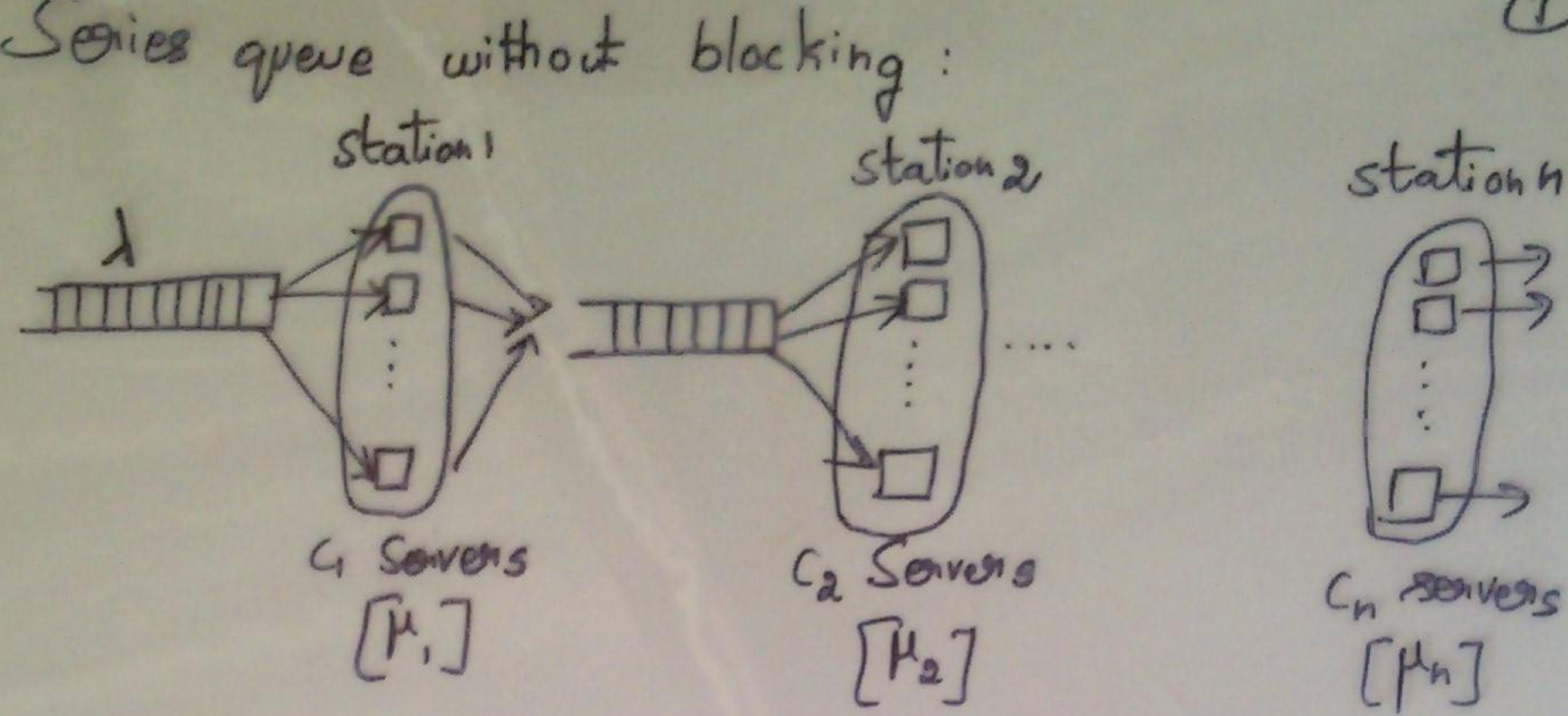
$$= \frac{\lambda^2 E(T^2) + 2\lambda E(T) [1 - \lambda E(T)]}{2[1 - \lambda E(T)]}$$

$$= \frac{\lambda^2 E(T^2)}{2[1 - \lambda E(T)]} + \lambda E(T)$$

$$= \frac{\lambda^2 \left[\sigma^2 + \frac{1}{\mu^2}\right]}{2\left[1 - \frac{\lambda}{\mu}\right]} + \frac{\lambda}{\mu}$$

$$\boxed{L_s = \frac{\lambda}{\mu} + \frac{\lambda^2\sigma^2 + \lambda^2/\mu^2}{2[1 - \lambda/\mu]}}$$

Series queue without blocking : ①




Burke's theorem:

Consider an $M/M/1$, $M/M/C$ or $M/M/\infty$ queueing system at steady state with poisson arrival rate $\lambda$, then

1. The departure process is Poisson with rate $\lambda$.
2. At each time $t$, the no of customers in the system is indepentent of the sequence of departure times prior to $t$.

## Equivalence property of queuing system:

(2)

If a service facility with c servers and an infinite queue has a poisson input with parameter $\lambda$ and the same exponential service time with parameter $\mu$ for each server of the M/M/c model, where $\frac{\lambda}{\mu c} < 1$ then, output of this facility is also a poisson process with rate $\lambda$ and this equals the input of the next facility.

**Remark:**

(1) Hence each station can be divided into an M/M/c model and hence a complete analysis of this type of series queue is possible.

(2) Product form solution :-

$$P_{n_1, n_2, \ldots n_m} = P_{n_1} P_{n_2} \ldots P_{n_m}$$

(3) Station with largest $\frac{\lambda}{\mu_i}$ is called the bottle neck of the system.

Two station tandem queue with single server each :- 3

WKT $\quad P[N_1 = n_1] = \left(\frac{\lambda}{\mu_1}\right)^{n_1}\left[1-\frac{\lambda}{\mu_1}\right] = \rho_1^{n_1}(1-\rho_1)$

$$P[N_2 = n_2] = \left(\frac{\lambda}{\mu_2}\right)^{n_2}\left[1-\frac{\lambda}{\mu_2}\right] = \rho_2^{n_2}\left[1-\rho_2\right]$$

$$\Rightarrow \quad P_{n_1,n_2} = \rho_1^{n_1}\rho_2^{n_2}(1-\rho_1)(1-\rho_2)$$

$\therefore$ Avg no of customers in the system $(L_s)$

$$= \sum_{n_1}\sum_{n_2}(n_1+n_2)P_{n_1,n_2}$$

$$= \sum_{n_1}\sum_{n_2} n_1\rho_1^{n_1}\rho_2^{n_2}(1-\rho_1)(1-\rho_2) + \sum_{n_1}\sum_{n_2} n_2\rho_1^{n_1}\rho_2^{n_2}(1-\rho_1)(1-\rho_2)$$

$$= (1-\rho_1)(1-\rho_2)\left\{\sum_{n_1} n_1\rho_1^{n_1}\sum_{n_2}\rho_2^{n_2} + \sum_{n_2} n_2\rho_2^{n_2}\sum_{n_1}\rho_1^{n_1}\right\}$$

$$= (1-\rho_1)(1-\rho_2)\left[\rho_1[1-\rho_1]^{-2}[1-\rho_2]^{-1} + \rho_2[1-\rho_2]^{-2}[1-\rho_1]^{-1}\right]$$

$$= (1-\rho_1)(1-\rho_2)(1-\rho_1)^{-1}(1-\rho_2)^{-1}\left\{\rho_1(1-\rho_1)^{-1} + \rho_2(1-\rho_2)^{-1}\right\}$$

$$= \frac{\rho_1}{1-\rho_1} + \frac{\rho_2}{1-\rho_2}$$

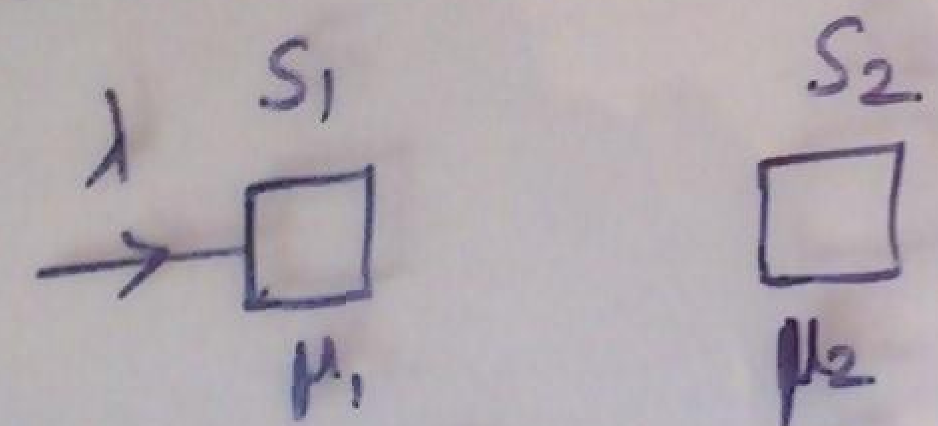
$$= L_{s_1} + L_{s_2}.$$

Using Little's formulae, $\quad W_s = W_{s_1} + W_{s_2}.$

Series Queue with blocking :-

Let us consider a two station sequential queue with single server at each of the stations $S_1$ & $S_2$ where no queue is allowed to be formed.

(ie) An entering customer has to goto $S_1$ for service and after completion of service will goto $S_2$ if it is empty, otherwise will wait at $S_1$ untill $S_2$ becomes empty (ie) The system is blocked.




$\{0, 1, b\}$ $\{0, 1\}$

Various states are $\{(0,0)\ (0,1)\ (1,0)\ (1,1)\ \cancel{(b,0)}\ (b,1)\}$

$$
\begin{array}{c|c}
 & \begin{array}{ccccc} (0,0) & (0,1) & (1,0) & (1,1) & (b,1) \end{array} \\
\begin{array}{c} (0,0) \\ (0,1) \\ (1,0) \\ (1,1) \\ (b,1) \end{array} &
\begin{bmatrix}
0 & 0 & \lambda & 0 & 0 \\
\mu_2 & 0 & 0 & \lambda & 0 \\
0 & \mu_1 & 0 & 0 & 0 \\
0 & 0 & \mu_2 & 0 & \mu_1 \\
0 & \mu_2 & 0 & 0 & 0
\end{bmatrix}
\end{array}
$$

Network of queues : ①

Network of queues may be described as a group of stations (or) nodes, where each node has $C_i$ servers.

What is the difference between series queue and network?

| Series Queue | Network |
|---|---|
| 1. Every customer enters the same station | Any customer can enter any station |
| 2. Every customer leave from the same station | Any customer may leave from any station |
| 3. Every customer goes from station to station in a sequencial order. | → Customers may visit stations already visited<br>→ They may skip some stations<br>→ They may even remain in the system forever. |

Effect:

Equivalence property and product form solution does not exist.

## Open and closed queueing network:-



→ A queueing network is said to be open if customers may enter from outside the system, circulate among the stations and then leave the system.

→ A queueing network is said to be closed if no customer may enter the system from outside and no customer may leave the system so that there is always a fixed number of customers in the network.

## Jackson network:-

[Special networks that satisfies product form solution]

Defn : A queueing network is called an open jackson network if it has the following characteristics.

(i) It is a system with m nodes, each node i has an infinite queue.

(ii) Customers may arrive from outside to node i follow poisson process with mean rate $r_i$

(iii) Service times at each channel at node i are independent and exponentially distributed with parameter $\mu_i$

(iv) A customer goes from node i to node j with prob $r_{ij}$ (or) leaves the system with prob $r_{i0}$

Also,

$$r_{i0} + \sum_{j=1}^{m} r_{ij} = 1$$

Traffic equations (or) flow balance equations:-

The total mean arrival rate at each node is,

$$\lambda_j = r_j + \sum_{i=1}^{m} \lambda_i r_{ij}$$

[ out + in ]

## Closed jackson network:-

Let N be the fixed no customers in a closed jackson network. Let $r_{ij}$ be the prob that a customer goes from node i to node j.

$$\Rightarrow \sum_{j=1}^{m} r_{ij} = 1$$

The traffic equations are,

$$\lambda_j = \sum_{i=1}^{m} \lambda_i r_{ij}$$

The steady state solution to these equations given by Jackson is,

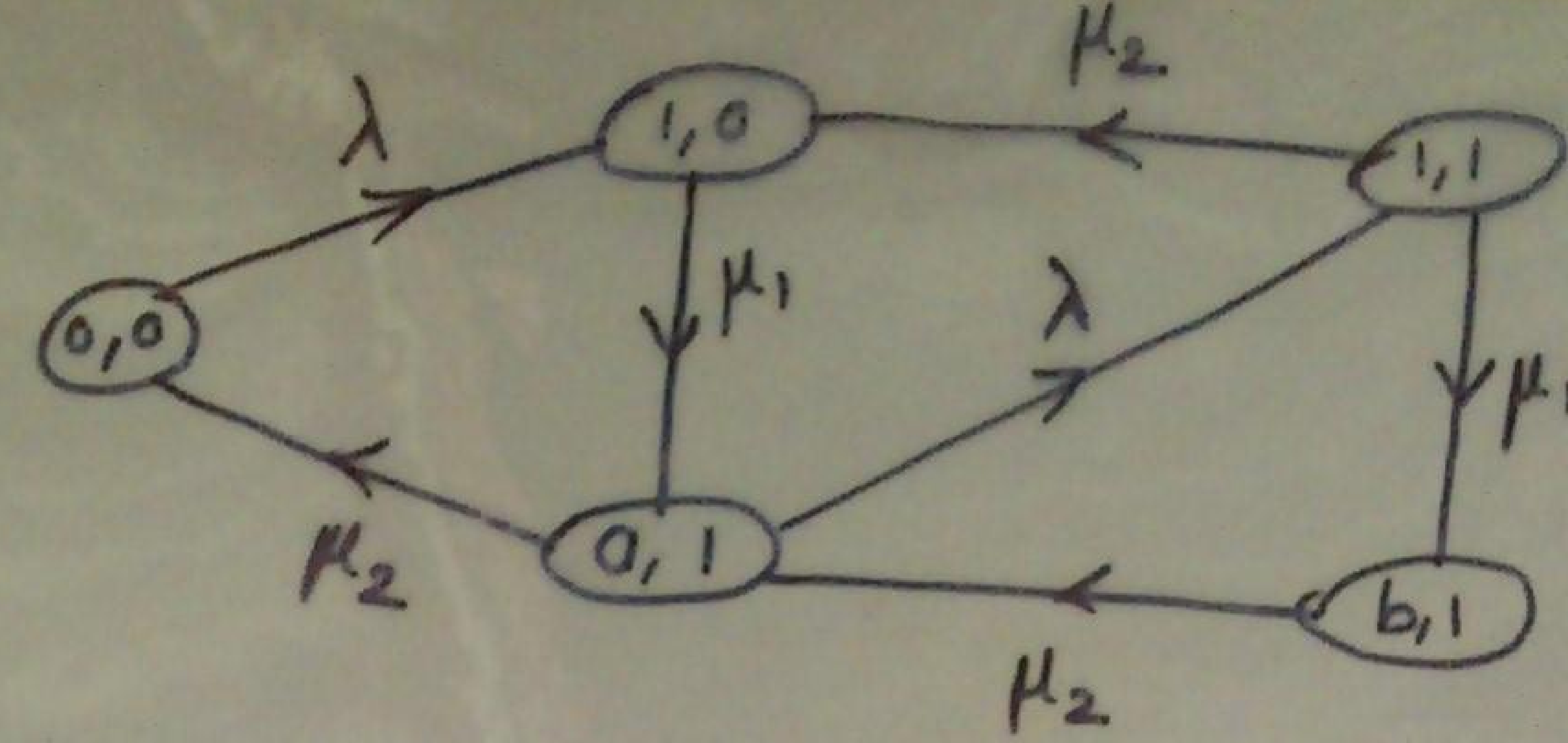
$$P_{n_1 n_2 \ldots n_m} = \begin{cases} K \rho_1^{n_1} \rho_2^{n_2} \cdots \rho_m^{n_3}, & \text{for single server at each node} \\ K \dfrac{\rho_1^{n_1}}{a_1(n_1)} \dfrac{\rho_2^{n_2}}{a_2(n_2)} \cdots \dfrac{\rho_m^{n_m}}{a_m(n_3)}, & \text{multiserver} \end{cases}$$

where $a_i(n_i) = \begin{cases} n_i! & n_i < c_i \\ c_i^{n_i - c_i} c_i! & n_i \geq c_i \end{cases}$

To find k:

$$\sum_{n_1+n_2+\cdots+n_m=N} P_{n_1 n_2 \ldots n_m} = 1$$

Remark:- Using these probabilities the entire system can be analysed.




1. Rate of each process = rate × prob from where it leaves.

2. At each node, Inrate = Outrate

Hence balance equations are,

$$\mu_2 P_{0,1} = \lambda P_{0,0}$$

$$\lambda P_{00} + \mu_2 P_{11} = \mu_1 P_{10}$$

$$\mu_1 P_{10} + \mu_2 P_{b1} = \mu_2 P_{01} + \lambda P_{01}$$

$$\lambda P_{01} = \mu_2 P_{11} + \mu_1 P_{11}$$

$$\mu_1 P_{11} = \mu_2 P_{b1}$$

Also $P_{00} + P_{10} + P_{01} + P_{11} + P_{b1} = 1$